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Title: Newtonian Hydrodynamics with Arbitrary Volumetric Sources

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1 Introduction

In this note, we derive how to handle mass, momentum, and energy sources for Newtonian hydrodynamics. Much of this is classic, although we're unaware of a reference that treats mass sources, necessary for certain physics and the method of manufactured solutions. In addition, we felt it important to emphasize that the integral form of the governing equations results in a straightforward treatment of the sources. With the integral form, we'll demonstrate that there's no ambiguity between the Lagrangian and Eulerian form of the equations, which is less clear with the differential forms.

2 Integral Equations

In this section, we review the integral equations for the conservation of mass, momentum, and total energy, for both the Lagrangian and Eulerian frames. These integral equations are general in the sense that they hold even if there are discontinuities in the solution. As we develop alternative differential forms below, it is critical that the integral form be in mind if conservation is to be attained.

2.1 Lagrangian Integral Equations

Consider a volume V [cm^3] that is moving with the fluid velocity $\vec{u}$ [cm/s]. The conservation of mass, momentum, and total energy in this Lagrangian volume can be summarized by the equations below:

- **Mass**:
$$\frac{D}{Dt}\int \rho\, dV = \int M\, dV\,, \tag{1}$$
where ρ [g/cm^3] is the fluid density and M [g/cm^3/s] is the mass volumetric source rate. Also,
$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u}\cdot\nabla\,. \tag{2}$$

- **Momentum**: Neglecting shear stresses, Newton's second law gives that the conservation of momentum is given by
$$\frac{D}{Dt}\int \rho\vec{u}\, dV + \oint p d\vec{A} = \int \left(M\vec{u} + \vec{F}\right) dV\,, \tag{3}$$
where $\vec{F}$ [force/cm^3/s] is the momentum volumetric source rate and p [force/cm^2] the pressure. The source $M\vec{u}$ is needed because even if $\vec{F} \equiv 0$, the momentum must change if the mass increases in the control volume.

- **Total Energy**: Neglecting heat conduction, the first-law of thermodynamics gives that the total energy (thermal + kinetic) obeys

$$\frac{D}{Dt}\int (\rho E)\; dV + \oint p\vec{u}\cdot d\vec{A} = \int \left(\frac{1}{2}Mu^2 + \vec{F}\cdot\vec{u} + \rho Q\right)\, dV\,, \tag{4}$$

where $E = e + \frac{1}{2}u^2$, Q [energy/g/s] is the specific energy source rate, and e [energy/g] is the specific internal energy. Note that

1. the term $\frac{1}{2}Mu^2$ represents the change in kinetic energy resulting from the mass source, M, and
2. the term $\vec{F}\cdot\vec{u}$ represents the work done on the control volume by $\vec{F}$.

2.2 Eulerian Integral Equations

Starting from the Lagrangian equations in the previous section, it's trivial to add the "flux terms" in the case that the volume V is stationary. We obtain

$$\frac{\partial}{\partial t}\int \rho\, dV + \oint \rho\vec{u}\cdot d\vec{A} = \int M\, dV\,, \tag{5}$$

$$\frac{\partial}{\partial t}\int \rho\vec{u}\, dV + \oint (\rho\vec{u}\vec{u} + p\mathbf{I})\, d\vec{A} = \int \left(M\vec{u} + \vec{F}\right) dV\,, \tag{6}$$

$$\frac{\partial}{\partial t}\int (\rho E)\, dV + \oint (\rho E\vec{u} + p\vec{u})\cdot d\vec{A} = \int \left(\frac{1}{2}Mu^2 + \vec{F}\cdot\vec{u} + \rho Q\right)\, dV\,, \tag{7}$$

where $\mathbf{I}$ is the identity tensor.

Note that these equations are very closely related to the Lagrangian equations. All that has changed is the time-derivative operator and the addition of flux terms on the boundary of the control volume. In particular, we note the following:

Remark. *The volumetric sources for mass, momentum, and total energy are identical for the Lagrangian and Eulerian reference frames for Netwonian hydrodynamics.*

This fact cannot be overemphasized. As a consequence, adding sources to either Lagrangian and Eulerian codes should be done in the same manner, consistent with the integral form of their respective equations. We note that for relativistic flows, the situation is more delicate and is left as an exercise for the reader.

3 Differential Equations

In this section, we will simplify the integral forms and derive several corresponding partial-differential equations (PDEs). We'll see that some of the PDEs are not necessarily obvious or intuitive, and I hope, emphasize the importance of starting any derivation with the integral form.

Below, we'll always start with the Lagrangian integral form. In order to simplify the derivations, we'll assume that the sources M, $\vec{F}$, and Q are constant in the control volume. Let also ρ, $\rho\vec{u}$, ρE represent average quantities over the control volume. The final PDEs will be rigorous, it's just that we've simplified the derivations.

3.1 Mass

With the assumptions above, the Lagrangian integral equation for mass, eq. (1), may be written as

$$\frac{D}{Dt}(\rho V) = MV\,. \tag{8}$$

Using the fact that

$$\frac{1}{V}\frac{DV}{Dt} = \nabla \cdot \vec{u}\,, \tag{9}$$

we have

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{u} = M\,. \tag{10}$$

Alternatively,

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{u}) = M\,, \tag{11}$$

which of course is the differential form of the Eulerian integral form (5).

3.2 Momentum

For momentum, (3) reduces to

$$\frac{D}{Dt}(V\rho\vec{u}) + \oint p d\vec{A} = \left(M\vec{u} + \vec{F}\right) V\,. \tag{12}$$

Using (9), dividing by V, and taking $V \to 0$, we obtain

$$\frac{D}{Dt}(\rho\vec{u}) + \rho\vec{u} \cdot \nabla\vec{u} + \nabla p = M\vec{u} + \vec{F}\,, \tag{13}$$

or

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \nabla \cdot (\rho\vec{u}\vec{u}) + \nabla p = M\vec{u} + \vec{F}\,. \tag{14}$$

This equation is the differential form of the Eulerian integral form (6).

Now, we can also expand the first term of (13) to obtain

$$\rho\frac{D\vec{u}}{Dt} + \vec{u}\frac{D\rho}{Dt} + \rho\vec{u} \cdot \nabla\vec{u} + \nabla p = M\vec{u} + \vec{F}\,, \tag{15}$$

Using (10), this equation reduces nicely to

$$\rho\frac{D\vec{u}}{Dt} + \nabla p = \vec{F}\,. \tag{16}$$

At this stage, it might be tempting to conclude that $\vec{F}$ is the Lagrangian or "comoving" momentum deposition. This is incorrect. Eq. (16) does not have a corresponding integral form; it does NOT represent the conservation of momentum. The volumetric source for (16) is the correct momentum depostion whenever $M \equiv 0$, which may be confusing for cases when $M \neq 0$.

3.3 Energy

For total energy, (4) reduces to

$$\frac{D}{Dt}(V\rho E) + \oint p\vec{u} \cdot d\vec{A} = \left(\frac{1}{2}Mu^2 + \vec{F} \cdot \vec{u} + \rho Q\right) V\,. \tag{17}$$

Using (9), dividing by V, and taking $V \to 0$, we obtain

$$\frac{D}{Dt}(\rho E) + \rho E\nabla \cdot \vec{u} + \nabla \cdot (p\vec{u}) = \frac{1}{2}Mu^2 + \vec{F} \cdot \vec{u} + \rho Q\,, \tag{18}$$

or

$$\frac{\partial}{\partial t}(\rho E) + \nabla \cdot (\rho E\vec{u} + p\vec{u}) = \frac{1}{2}Mu^2 + \vec{F} \cdot \vec{u} + \rho Q\,, \tag{19}$$

which is the differential form of (7).

We can also derive other Lagrangian forms from (18). Using (10),

$$\rho \frac{DE}{Dt} + \nabla \cdot (p\vec{u}) = \vec{F} \cdot \vec{u} + \rho Q - Me\,. \tag{20}$$

Again, it may be tempting to identify the right-hand side as the Lagrangian or "comoving" total energy deposition. But again, (20) does NOT represent the conservation of total energy, in any frame. Similar to the momentum case in the previous section, note that that the volumetric source for (20) is the correct energy deposition whenever $M \equiv 0$. Again, we believe this may cause confusion when $M \neq 0$.

We can also write a Lagrangian equation for the internal energy from (20), as

$$\rho \frac{De}{Dt} + p\nabla \cdot u = \rho Q - Me\,. \tag{21}$$

This is apparently a 2nd-law of thermodynamics, but with mass change. Even with $M \equiv 0$, it's not a statement of the convervation of energy.